

THEORY OF VALUATIONS ON MANIFOLDS, I. LINEAR SPACES

BY

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ABSTRACT

This is the first part of a series of articles where we are going to develop theory of valuations on manifolds generalizing the classical theory of continuous valuations on convex subsets of an affine space. In this article we still work only with linear spaces. We introduce a space of smooth (non-translation invariant) valuations on a linear space V . We present three descriptions of this space. We describe the canonical multiplicative structure on this space generalizing the results from [4] obtained for polynomial valuations.

0. Introduction

This is the first part of a series of articles where we are going to develop theory of valuations on manifolds generalizing the classical theory of continuous valuations on convex subsets of an affine space. In this article we still work only with linear spaces. In the subsequent parts of this series we are going to generalize constructions of this article to arbitrary smooth manifolds [5], [6]. The case of a linear space considered here will be useful for the general case for technical reasons.

Let us recall some basic definitions. Let V be a finite dimensional real vector space, $n = \dim V$. Let $\mathcal{K}(V)$ denote the class of all convex compact subsets of V . Equipped with the Hausdorff metric, the space $\mathcal{K}(V)$ is a locally compact space.

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0.1.1 Definition: (a) A function $\phi: \mathcal{K}(V) \rightarrow \mathbb{C}$ is called a valuation if for any $K_1, K_2 \in \mathcal{K}(V)$ such that their union is also convex one has

$$\phi(K_1 \cup K_2) = \phi(K_1) + \phi(K_2) - \phi(K_1 \cap K_2).$$

(b) A valuation ϕ is called continuous if it is continuous with respect to the Hausdorff metric on $\mathcal{K}(V)$.

For the classical theory of valuations we refer to the surveys McMullen–Schneider [16] and McMullen [15]. For the general background from convexity we refer to Schneider [17].

In this article we introduce the space $SV(V)$ of *smooth* valuations on V (see Definition 2.1.2). $SV(V)$ is a Fréchet space. We present three different descriptions of this space. We describe a canonical structure of commutative associative topological algebra with unit (where the unit is the Euler characteristic).

Moreover, the algebra $SV(V)$ has a canonical filtration by closed subspaces

$$SV(V) = W_0 \supset W_1 \supset \cdots \supset W_n$$

compatible with the product, namely $W_i \cdot W_j \subset W_{i+j}$. Note that the subspace W_n coincides with the space of smooth densities on V . Moreover, in Theorem 4.1.3 we prove that there exists a canonical isomorphism of the associated graded algebra $gr_W SV(V) := \bigoplus_{i=0}^n W_i/W_{i+1}$ and the algebra $C^\infty(V, Val^{sm}(V))$ of infinitely smooth functions on V with values in the algebra $Val^{sm}(V)$ of smooth translation invariant valuations.

Note also that the space $SV(V)$ contains polynomial smooth valuations (studied by the author in [4]) as a dense subspace. Thus the above results generalize results on polynomial valuations from [4].

The paper is organized as follows. In Section 1 we recall some necessary facts from representation theory (Subsection 1.1) and valuation theory (Subsection 1.2).

In Section 2 we introduce the main object of this article, namely the space of smooth valuations (Definition 2.1.2).

In Section 3 we introduce the filtration W_\bullet on $SV(V)$. We study its basic properties. In particular, we show in Proposition 3.1.5 the isomorphism of Fréchet spaces

$$gr_W SV(V) \simeq C^\infty(V, Val^{sm}(V)).$$

This isomorphism is the first description of the space of smooth valuations. In Corollary 3.1.7 we obtain the second one.

In Section 4 we introduce and study the canonical multiplicative structure on $SV(V)$.

In Section 5 we recall the construction of continuous valuations using integration with respect to the normal cycle. Then we show in Theorem 5.2.2 that all smooth valuations are obtained in this way. This is the third promised description of smooth valuations.

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1. Background

In this section we recall some necessary facts from representation theory (Subsection 1.1) and theory of valuations (Subsection 1.2). No result of this section is new.

1.1 SOME REPRESENTATION THEORY.

1.1.1 Definition: Let ρ be a continuous representation of a Lie group G in a Fréchet space F . A vector $\xi \in F$ is called G -smooth if the map $g \mapsto \rho(g)\xi$ is an infinitely differentiable map from G to F .

It is well known (see e.g. [18], Section 1.6) that the subset F^{sm} of smooth vectors is a G -invariant linear subspace dense in F . Moreover, it has a natural topology of a Fréchet space (which is stronger than the topology induced from F), and the representation of G in F^{sm} is continuous. Moreover, all vectors in F^{sm} are G -smooth.

Let G be a real reductive group. Assume that G can be imbedded into the group $GL_N(\mathbb{R})$ for some N as a closed subgroup invariant under the transposition. Let us fix such an imbedding $p: G \hookrightarrow GL_N(\mathbb{R})$. (In our applications G will be either $GL_n(\mathbb{R})$ or a direct product of several copies of $GL_n(\mathbb{R})$.) Let us introduce a norm $|\cdot|$ on G as follows:

$$|g| := \max\{\|p(g)\|, \|p(g^{-1})\|\}$$

where $\|\cdot\|$ denotes the usual operator norm in \mathbb{R}^N .

1.1.2 Definition: Let (π, G, F) be a smooth representation of G in a Fréchet space F (namely $F^{sm} = F$). One says that this representation has **moderate growth** if for each continuous semi-norm λ on F there exists a continuous semi-norm ν_λ on F and $d_\lambda \in \mathbb{R}$ such that

$$\lambda(\pi(g)v) \leq \|g\|^{d_\lambda} \nu_\lambda(v)$$

for all $g \in G, v \in F$.

The proof of the next lemma can be found in [18], Lemmas 11.5.1 and 11.5.2.

1.1.3 LEMMA: (i) If (π, G, H) is a continuous representation of G in a Banach space H then (π, G, H^{sm}) has moderate growth.

(ii) Let (π, G, V) be a representation of moderate growth. Let W be a closed G -invariant subspace of V . Then W and V/W have moderate growth.

The next lemma is obvious.

1.1.4 LEMMA: Let G_1 be a closed reductive subgroup of a reductive group G . Assume that the image of G_1 in $GL_N(\mathbb{R})$ under the map $p: G \hookrightarrow GL_N(\mathbb{R})$ is closed under the transposition. Let (π, G, F) have moderate growth. Then the restriction of this representation to G_1 also has moderate growth.

Recall that a continuous Fréchet representation (ρ, G, \mathcal{F}) is said to have **finite length** if there exists a finite filtration

$$0 = F_0 \subset F_1 \subset \dots \subset F_m = F$$

by G -invariant closed subspaces such that F_i/F_{i-1} is irreducible, i.e. does not have proper closed G -invariant subspaces.

A Fréchet representation (ρ, G, F) of a real reductive group G is called **admissible** if its restriction to a maximal compact subgroup K of G contains an isomorphism class of any irreducible representation of K with at most finite multiplicity. (Recall that a maximal compact subgroup of $GL_n(\mathbb{R})$ is the orthogonal group $O(n)$.)

1.1.5 THEOREM (Casselman–Wallach, [7]): Let G be a real reductive group. Let (ρ, G, F_1) and (π, G, F_2) be smooth representations of moderate growth in Fréchet spaces F_1, F_2 . Assume in addition that F_2 is admissible of finite length. Then any continuous morphism of G -modules $f: F_1 \rightarrow F_2$ has closed image.

Let us also recall the classical L. Schwartz kernel theorem.

1.1.6 THEOREM (L. Schwartz kernel theorem, [9]): *Let X_1 and X_2 be compact smooth manifolds. Let \mathcal{E}_1 and \mathcal{E}_2 be smooth finite dimensional vector bundles over X_1 and X_2 respectively. Let \mathcal{G} be a Fréchet space. Let*

$$B: C^\infty(X_1, \mathcal{E}_1) \times C^\infty(X_2, \mathcal{E}_2) \rightarrow \mathcal{G}$$

be a continuous bilinear map. Then there exists a unique continuous linear operator

$$b: C^\infty(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2) \rightarrow \mathcal{G}$$

such that $b(f_1 \otimes f_2) = B(f_1, f_2)$ for any $f_i \in C^\infty(X_i, \mathcal{E}_i), i = 1, 2$.

The proof of the L. Schwartz kernel theorem is based on the next elementary and well known lemma which will be used in this article.

1.1.7 LEMMA: *Let X_1, X_2 be two smooth manifolds such that X_2 is compact. Let \mathcal{E}_1 and \mathcal{E}_2 be smooth finite dimensional vector bundles over X_1 and X_2 respectively. Let $M \in \mathbb{N}$ be an integer. Let $G \subset X_1$ be a compact subset.*

Then there exists a compact subset $\tilde{G} \subset X_1$ containing G , an integer $N \in \mathbb{N}$, and a constant C such that for any $f \in C^\infty(X_1 \times X_2, \mathcal{E}_1 \boxtimes \mathcal{E}_2)$ there exists a presentation

$$f = \sum_{i=1}^{\infty} g_i \otimes h_i$$

such that $g_i \in C^\infty(X_1, \mathcal{E}_1), h_i \in C^\infty(X_2, \mathcal{E}_2)$ and

$$\sum_{i=1}^{\infty} \|g_i\|_{C^M(G)} \|h_i\|_{C^M(X_2)} \leq C \|f\|_{C^N(\tilde{G} \times X_2)}.$$

For a Fréchet space F and a smooth manifold X let us denote by $C^\infty(X, F)$ the Fréchet space of infinitely smooth F -valued functions on X with the topology of uniform convergence with all derivatives on compact subsets of X . The next proposition is well known but we do not have a reference.

1.1.8 PROPOSITION: *Let G be a real reductive Lie group. Let F_1, F_2 be continuous Fréchet G -modules. Let $\xi: F_1 \rightarrow F_2$ be a continuous morphism of G -modules. Assume that the assumptions of the Casselman–Wallach theorem are satisfied, namely F_1 and F_2 are smooth and have moderate growth, and F_2 is admissible of finite length. Assume moreover that ξ is surjective.*

Let X be a smooth manifold. Consider the map

$$\hat{\xi}: C^\infty(X, F_1) \rightarrow C^\infty(X, F_2)$$

defined by $(\hat{\xi}(f))(x) = \xi(f(x))$ for any $x \in X$.

Then $\hat{\xi}$ is surjective.

Proof: First let us prove this proposition under the assumption that $X = \mathbb{P}^n$. Consider the natural action of the group $GL_{n+1}(\mathbb{R})$ on \mathbb{P}^n . Then the representation of the reductive group $G \times GL_{n+1}(\mathbb{R})$ in the spaces $C^\infty(X, F_i), i = 1, 2$ is smooth and of moderate growth. Moreover, $C^\infty(X, F_2)$ is an admissible $(G \times GL_{n+1}(\mathbb{R}))$ -module of finite length. Hence by the Casselman–Wallach theorem it is enough to show that $\hat{\xi}$ has dense image. But since ξ is surjective, the image of $\hat{\xi}$ contains finite linear combinations of elements of the form $f \otimes v$ with $f \in C^\infty(\mathbb{P}^n), v \in F_2$. Clearly such linear combinations are dense in $C^\infty(X, F_2)$.

Let us return to the case of a general manifold X . Let us denote by \mathcal{O}_X the sheaf of infinitely smooth functions on X . Let us consider the sheaves $\mathcal{F}_i, i = 1, 2$, on X defined by

$$\mathcal{F}_i(U) = C^\infty(U, F_i)$$

for any open subset $U \subset X$. Then ξ induces a morphism of \mathcal{O}_X -modules (which will be also denoted by $\hat{\xi}$)

$$\hat{\xi}: \mathcal{F}_1 \rightarrow \mathcal{F}_2$$

which is obviously defined. Let us show that $\hat{\xi}$ is an epimorphism of sheaves. Since this statement is local, and all smooth manifolds of given dimension are locally diffeomorphic, we may assume again that $X = \mathbb{P}^n$. Fix a point $x \in \mathbb{P}^n$. Let $B_1 \subset B_2 \subset \mathbb{P}^n$ be two open balls such that $x \in B_1$ and the closure of B_1 is contained in B_2 . Let us fix a function $\gamma \in C^\infty(\mathbb{P}^n)$ such that $\gamma|_{B_1} \equiv 1, \gamma|_{\mathbb{P}^n \setminus B_2} \equiv 0$. Let $\phi \in H^0(B_2, \mathcal{F}_2)$. Set $\psi := \gamma \cdot \phi$. Then ψ extends by zero to a section from $H^0(\mathbb{P}^n, \mathcal{F}_2) = F_2$. This section will be denoted again by ψ . By the previous case there exists $\chi \in F_1 = H^0(\mathbb{P}^n, \mathcal{F}_1)$ such that $\hat{\xi}(\chi) = \psi$. Restricting the last identity to B_1 we conclude that $\hat{\xi}$ is an epimorphism of sheaves.

For a general X , let us denote $\mathcal{K} := Ker \hat{\xi}$. Thus \mathcal{K} is an \mathcal{O}_X -module. It is well known that every \mathcal{O}_X -module \mathcal{K} is acyclic, i.e. $H^i(X, \mathcal{K}) = 0$ for $i > 0$ (see e.g. [10]).

We have a short exact sequence of sheaves

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow 0.$$

From the long exact sequence we get

$$H^0(X, \mathcal{F}_1) \rightarrow H^0(X, \mathcal{F}_2) \rightarrow H^1(X, \mathcal{K}) = 0.$$

The result is proved. ■

1.1.9 Remark: Recently we were informed by B. Mityagin and V. Palamodov that the following general fact, which is also sufficient for our purposes instead of Proposition 1.1.8, is true. Let $\xi: F_1 \rightarrow F_2$ be a surjective continuous linear map of nuclear Fréchet spaces. Let X be a smooth manifold. Then $\hat{\xi}: C^\infty(X, F_1) \rightarrow C^\infty(X, F_2)$ is surjective.

1.2 SOME VALUATION THEORY. Let us recall a few basic facts from the theory of translation invariant continuous valuations. For a real vector space V of finite dimension n let us denote by $Val(V)$ the space of translation invariant valuations on $\mathcal{K}(V)$ continuous with respect to the Hausdorff metric on $\mathcal{K}(V)$. Equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$ the space $Val(V)$ becomes a Banach space (see e.g. Lemma A.4 in [4]).

1.2.1 Definition: A valuation ϕ is called homogeneous of degree k (or just k -homogeneous) if for every convex compact set K and for every scalar $\lambda > 0$

$$\phi(\lambda K) = \lambda^k \phi(K).$$

Let us denote by $Val_k(V)$ the space of translation invariant continuous valuations homogeneous of degree k .

1.2.2 THEOREM (McMullen [14]):

$$Val(V) = \bigoplus_{k=0}^n Val_k(V),$$

where $n = \dim V$.

Note in particular that the degree of homogeneity is an integer between 0 and $n = \dim V$. It is known that $Val_0(V)$ is one-dimensional and is spanned by the Euler characteristic χ , and $Val_n(V)$ is also one-dimensional and is spanned by a Lebesgue measure [11]. The space $Val_n(V)$ is also denoted by $|\wedge V^*|$ (the space of complex valued Lebesgue measures on V). One has further decomposition with respect to parity:

$$Val_k(V) = Val_k^{ev}(V) \oplus Val_k^{odd}(V),$$

where $Val_k^{ev}(V)$ is the subspace of even valuations (ϕ is called even if $\phi(-K) = \phi(K)$ for every $K \in \mathcal{K}(V)$), and $Val_k^{odd}(V)$ is the subspace of odd valuations (ϕ is called odd if $\phi(-K) = -\phi(K)$ for every $K \in \mathcal{K}(V)$). The Irreducibility Theorem is as follows.

1.2.3 THEOREM (Irreducibility Theorem [3]): *The natural representation of the group $GL(V)$ on each space $Val_k^{ev}(V)$ and $Val_k^{odd}(V)$ is irreducible for any $k = 0, 1, \dots, n$.*

In this theorem, by the natural representation one means the action of $g \in GL(V)$ on $\phi \in Val(V)$ as $(g\phi)(K) = \phi(g^{-1}K)$ for every $K \in \mathcal{K}(V)$. The subspace of smooth valuations with respect to this action in the sense of Definition 1.1.1 is denoted by $Val^{sm}(V)$.

1.2.4 Remark: The representation $Val(V)$ of $GL(V)$ is an admissible representation. Indeed, it was show in [2] that $Val_k^{ev/odd}$ can be $GL(V)$ -equivariantly imbedded into the space of continuous sections of a $GL(V)$ -equivariant finite dimensional vector bundle $\mathcal{E}^{ev/odd}$ over the projective space $\mathbb{P}_+(V^*)$. Let us show that this representation must be admissible. Let us fix on V^* a Euclidean metric. Let us fix a point $l_0 \in \mathbb{P}(V^*)$. Let H denote the stabilizer of l_0 in $O(n)$. Then $\mathbb{P}_+(V^*) \simeq O(n)/H$. Let $q: O(n) \rightarrow \mathbb{P}(V^*)$ be the surjection $g \mapsto g(l_0)$. Let $\mathcal{L} := q^*\mathcal{E}^{ev/odd}$. Then \mathcal{L} is an $O(n)$ -equivariant vector bundle over $O(n)$, hence \mathcal{L} is $O(n)$ -equivariantly trivial. Note that we have the $O(n)$ -equivariant imbedding $C(\mathbb{P}(V^*)) \hookrightarrow C(O(n), \mathcal{L})$. Hence it is enough to check that $C(O(n), \mathcal{L})$ contains each irreducible representation of $O(n)$ with at most finite multiplicity. This follows from the well known fact that for any compact group K the space of functions $C(K)$ contains each irreducible representation π of K with multiplicity $\dim \pi$ (which is necessarily finite).

2. The space of smooth valuations

2.1 SOME DEFINITIONS. Let V be an n -dimensional real vector space. Let us denote by $CV(V)$ the space of continuous valuations on V . Equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$, $CV(V)$ becomes a Fréchet space. Let $QV(V)$ denote the space of continuous valuations on V which satisfy the following additional property:

the map given by $K \mapsto \phi(tK + x)$ is a continuous map $\mathcal{K}(V) \rightarrow C^n([0, 1] \times V)$. Let us call such valuations **quasi-smooth**.

In the space $QV(V)$ we have the natural linear topology defined as follows. Fix a compact subset $G \subset V$. Define a seminorm on $QV(V)$

$$\|\phi\|_G := \sup\{\|\phi(tK + x)\|_{C^n([0,1] \times G)} \mid K \subset G\}.$$

Note that the seminorm $\|\cdot\|_G$ is finite. One easily checks the following claim.

2.1.1 CLAIM: *Equipped with the topology defined by this sequence of seminorms the space $QV(V)$ is a Fréchet space.*

Note also that the natural representation of the group $Aff(V)$ of affine transformations of V in the space $QV(V)$ is continuous. We will denote by $SV(V)$ the subspace of $Aff(V)$ -smooth vectors in $QV(V)$. It is a Fréchet space.

2.1.2 Definition: Elements of $SV(V)$ will be called **smooth** valuations on V .

2.2 MAIN EXAMPLES. Let V be a real vector space of dimension n . Let us denote by $\mathbb{P}_+(V^*)$ the manifold of oriented lines passing through the origin in V^* . Let L denote the line bundle over $\mathbb{P}_+(V^*)$ such that its fiber over an oriented line l consists of linear functionals on l . Let $|\omega_V|$ denote the line bundle of densities over V . Let $p: V \times \mathbb{P}_+(V^*) \rightarrow V$ be the projection. For any integer $k, 0 \leq k \leq n$, we are going to construct a natural map

$$\Theta_k: \bigoplus_{j=0}^k C^\infty(V \times (\mathbb{P}_+(V^*))^j, |\omega_V| \boxtimes L^{\boxtimes j}) \rightarrow SV(V).$$

First let us recall some results from [1]. Let $\bar{K} = (K_1, K_2, \dots, K_s)$ be an s -tuple of compact convex subsets of V . Let $r \in \mathbb{N} \cup \{\infty\}$. For any $\mu \in C^r(V, |\omega_V|)$ consider the function $M_{\bar{K}}\mu: \mathbb{R}_+^s \rightarrow \mathbb{C}$, where $\mathbb{R}_+^s = \{(\lambda_1, \dots, \lambda_s) \mid \lambda_j \geq 0 \text{ for all } j\}$, defined by

$$(M_{\bar{K}}\mu)(\lambda_1, \dots, \lambda_s) = \mu\left(\sum_{j=1}^s \lambda_j K_j\right).$$

2.2.1 THEOREM ([1]): (1) $M_{\bar{K}}\mu \in C^r(\mathbb{R}_+^s)$ and $M_{\bar{K}}$ is a continuous operator from $C^r(V, |\omega_V|)$ to $C^r(\mathbb{R}_+^s)$.

(2) Assume that a sequence $\mu^{(m)}$ converges to μ in $C^r(V, |\omega_V|)$. Let $K_j^{(m)}, K_j, j = 1, \dots, s, m \in \mathbb{N}$, be convex compact sets in V , and for every $j = 1, \dots, s, K_j^{(m)} \rightarrow K_j$ in the Hausdorff metric as $m \rightarrow \infty$. Then $M_{\bar{K}^{(m)}}\mu^{(m)} \rightarrow M_{\bar{K}}\mu$ in $C^r(\mathbb{R}_+^s)$.

Before we define the map Θ_k let us make some more remarks. Fix $s, 0 \leq s \leq k$. Let us fix $\mu \in C^\infty(V, |\omega_V|)$ and $A_1, \dots, A_s \in \mathcal{K}(V)$ being strictly convex with smooth boundaries. Let us define

$$(\Theta'_s(\mu; A_1, \dots, A_s))(K) := \frac{\partial^s}{\partial \lambda_1 \dots \partial \lambda_s} \Big|_0 \mu\left(K + \sum_{j=1}^s \lambda_j A_j\right).$$

Theorem 2.2.1 implies that $\Theta'_s(\mu; A_1, \dots, A_s) \in SV(V)$. It is clear that Θ'_s is Minkowski additive with respect to each A_j . Namely, say for $j = 1$, one has

$\Theta'_s(\mu; aA'_1 + bA''_1, A_2, \dots, A_s) = a\Theta'_s(\mu; A'_1, A_2, \dots, A_s) + b\Theta'_s(\mu; A''_1, A_2, \dots, A_s)$
 for $a, b \geq 0$.

Recall that for any $A \in \mathcal{K}(V)$ one defines the supporting functional $h_A(y) := \sup_{x \in A} y(x)$ for any $y \in V^*$. Thus $h_A \in C(\mathbb{P}_+(V^*), L)$. Moreover, it is well known (and easy to see) that $A_N \rightarrow A$ in the Hausdorff metric if and only if $h_{A_N} \rightarrow h_A$ in $C(\mathbb{P}_+(V^*), L)$. Also, any section $F \in C^2(\mathbb{P}_+(V^*), L)$ can be presented as a difference $F = G - H$ where $F, H \in C^2(\mathbb{P}_+(V^*), L)$ are supporting functionals of some convex compact sets and $\max\{\|G\|_2, \|H\|_2\} \leq c\|F\|_2$, where a constant c is independent of F . (Indeed one can choose $G = F + R \cdot h_D, H = R \cdot h_D$ where D is the unit Euclidean ball, and R is a large enough constant depending on $\|F\|_2$.) Hence we can uniquely extend Θ'_s to a multilinear continuous map (which we will denote by the same letter):

$$\Theta'_s: C^\infty(V, |\omega_V|) \times (C^\infty(\mathbb{P}_+(V^*), L))^s \rightarrow SV(V).$$

Theorem 2.2.1 implies that Θ'_s depends continuously on each argument. By the L. Schwartz kernel theorem it follows that this map gives rise to a continuous linear map

$$\Theta'_s: C^\infty(V \times \mathbb{P}_+(V^*)^s, |\omega_V| \boxtimes L^{\boxtimes s}) \rightarrow SV(V).$$

Now let us define the map

$$\Theta_k := \bigoplus_{i=0}^k \Theta'_i.$$

3. Filtrations

Let us define a decreasing filtration on $SV(V)$. Set

$$W_i := \left\{ \phi \in SV(V) \mid \frac{d^k}{dt^k} \phi(tK + x)|_{t=0} = 0 \forall k < i, K \in \mathcal{K}(V), x \in V \right\}.$$

It is clear that W_i are $Aff(V)$ -invariant closed subspaces of $SV(V)$. Obviously $SV(V) = W_0 \supset W_1 \supset \dots$.

3.1.1 PROPOSITION:

$$W_{n+1} = 0.$$

Proof: Let $\phi \in W_{n+1}$. We want to show that ϕ vanishes. Let us prove it by induction in $n = \dim V$. For $n = 0$ the statement is clear. Let us assume that the statement holds for $n - 1$. Then this implies that ϕ is a simple valuation, i.e. it vanishes on convex sets of dimension less than n . It is sufficient to show that ϕ vanishes on polytopes. Since every polytope can be dissected into simplices, it

is sufficient to prove that ϕ vanishes on simplices. Let Δ be a simplex. Choosing an appropriate coordinate system we may assume that it has the form

$$\Delta = \{(x_1, \dots, x_n) | 0 \leq x_1 \leq \dots \leq x_n \leq 1\}.$$

Set for $1 \leq i \leq j \leq n$,

$$T_{i,j} :=$$

$$\{(x_1, \dots, x_n) | 0 \leq x_i \leq x_{i+1} \leq \dots \leq x_j \leq 1 \text{ and } x_l = 0 \text{ for } l < i \text{ and } l > j\}.$$

For a sequence $0 < j_1 < \dots < j_{l-1} < n$, let us denote (as in [14])

$$T_{j_1 \dots j_{l-1}} := T_{0j_1} + \dots + T_{j_{l-1}, n}.$$

First note that any point $z \in \Delta$ has the form $z = (z_i)_{i=1}^n$, where

$$(1) \quad z_1 = \dots = z_{j_1} < z_{j_1+1} = \dots = z_{j_2} < \dots < z_{j_{l-1}+1} = \dots = z_{j_l} \leq 1,$$

and $j_l = n$.

For a sequence $0 < j_1 < \dots < j_{l-1} < n$ let us also define

$$R_{j_1 \dots j_{l-1}}(N) := \left\{ z \in \frac{1}{N} \mathbb{Z}^n \cap \Delta \mid z \text{ satisfies (1)} \right\}.$$

Then since ϕ is a simple valuation one has

$$(2) \quad \phi(\Delta) = \sum_{0 < j_1 < \dots < j_{l-1} < n} \left(\sum_{z \in R_{j_1 \dots j_{l-1}}(N)} \phi\left(z + \frac{1}{N} T_{j_1 \dots j_{l-1}}\right) \right).$$

Since $\phi \in W_{n+1}$, for any $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $N > N(\varepsilon)$ and for all $z \in \Delta$ one has $|\phi(z + \frac{1}{N} T_{j_1 \dots j_{l-1}})| < \varepsilon N^{-n}$. However, it is easy to see that $|\frac{1}{N} \mathbb{Z}^n \cap \Delta| \leq CN^n$ where C is a constant independent of N . From equation (2) we get an estimate $|\phi(\Delta)| \leq C' \varepsilon$ where C' is a constant depending on n only. Hence $\phi(\Delta) = 0$. Hence $\phi \equiv 0$. ■

3.1.2 PROPOSITION: W_n coincides with the space of smooth densities on V .

Proof: Obviously smooth densities are contained in W_n . Now let us fix $\phi \in W_n$. Let $\tilde{\phi}(K, x) = \frac{d^n}{dt^n} |_{t=0} \phi(tK + x)$. Then for a fixed $x \in V$, $\tilde{\phi}(\cdot, x)$ is a translation invariant continuous valuation homogeneous of degree n . To check it, fix an arbitrary compact subset $G \subset V$. We have

$$\phi(tK + x) = \frac{t^n}{n!} \tilde{\phi}(K, x) + o(t^k)$$

uniformly on $K \subset G, t \in G$. Also $\tilde{\phi}(K, x)$ depends smoothly on x when K is fixed. Then $\tilde{\phi}(t(K+a)+x) = \frac{t^n}{n!} \tilde{\phi}(K, ta+x) + o(t^n) = \frac{t^n}{n!} \tilde{\phi}(K, x) + o(t^n)$. Hence $\tilde{\phi}(K, x)$ is translation invariant in K when x is fixed.

By a result due to Hadwiger [11] $\tilde{\phi}(\cdot, x)$ must be a Lebesgue measure. Also it depends smoothly on $x \in V$. Subtracting from ϕ an appropriate density and using the fact that $W_{n+1} = 0$ by Proposition 3.1.1, we deduce the result. ■

3.1.3 Example: (1) Let us recall the definition of a **polynomial valuation** introduced by Khovanskii and Pukhlikov [13]. A valuation ϕ is called polynomial of degree at most d if for any $K \in \mathcal{K}(V)$ the function $V \rightarrow \mathbb{C}$ given by $x \mapsto \phi(K+x)$ is a polynomial of degree at most d . In [13] it was shown that if ϕ is a continuous polynomial valuation of degree d , then for any $K_1, \dots, K_s \in \mathcal{K}(V)$ the function $\phi(\sum_j \lambda_j K_j)$ is a polynomial in $\lambda_j \geq 0$ of degree at most $d+n$. It follows that $\phi \in QV(V)$.

(2) Let μ be a smooth density. It was shown in [1] that the map

$$(K_1, \dots, K_s; \lambda_1, \dots, \lambda_s) \mapsto \mu\left(\sum_{j=1}^s \lambda_j K_j\right)$$

defines a continuous map $\mathcal{K}(V)^s \rightarrow C^\infty(\mathbb{R}_+^s)$. It follows that for any fixed $A_1, \dots, A_s \in \mathcal{K}(V)^s$ the map

$$K \mapsto \frac{\partial^s}{\partial \lambda_1 \dots \partial \lambda_s} \Big|_0 \mu\left(K + \sum_j \lambda_j A_j\right)$$

defines a valuation from $QV(V)$.

Recall that in Section 2.2 we defined a map

$$\Theta_k: \bigoplus_{i=0}^k C^\infty(V \times \mathbb{P}_+(V^*)^i, |\omega_V| \boxtimes L^{\boxtimes i}) \rightarrow SV(V).$$

3.1.4 PROPOSITION: *The image of Θ_k is contained in W_{n-k} .*

Proof: Indeed $\mu(rK+x+\sum_j \lambda_j A_j) = O((\sqrt{r^2+\sum_j \lambda_j^2})^n)$. The result follows from the construction of Θ_k . ■

In Corollary 3.1.7 we will prove that in fact the image of Θ_k coincides with W_{n-k} . Let us denote by $Val(TV)$ the (infinite dimensional) vector bundle over V whose fiber over $x \in V$ is equal to the space of translation invariant

$GL(T_x V)$ -smooth valuations on the tangent space $T_x V$. Similarly, we can define the vector bundle $Val_k(TV)$ of k -homogeneous smooth translation invariant valuations. Clearly, $C^\infty(V, Val_k(TV)) = C^\infty(V, Val_k^{sm}(V))$ where the last space denotes the space of C^∞ -smooth functions on V with values in the Fréchet space $Val_k^{sm}(V)$ of k -homogeneous translation invariant smooth valuations.

Let us define a map

$$\Lambda_k: W_k \rightarrow C^\infty(V, Val_k(TV))$$

by $\Lambda_k(\phi) := [K \mapsto [x \mapsto \frac{1}{k!} \frac{d^k}{dt^k} |_{t=0} \phi(tK + x)]]$.

3.1.5 PROPOSITION: (i) $\Lambda_k: W_k \rightarrow C^\infty(V, Val_k(TV))$ is an epimorphism.

(ii) $Ker \Lambda_k = W_{k+1}$.

(iii) W_k/W_{k+1} is isomorphic to $C^\infty(V, Val_k(TV))$.

Proof: Clearly (iii) follows from (i) and (ii). Part (ii) is obvious from the definitions.

Let us check next that $K \mapsto \frac{1}{k!} \frac{d^k}{dt^k} |_{t=0} \phi(tK + x)$ is a translation invariant continuous valuation for any $x \in V$, and it depends smoothly on x . The only thing one should check is the translation invariance. Let us denote $\psi(K, x) := \frac{1}{k!} \frac{d^k}{dt^k} |_{t=0} \phi(tK + x)$. Then for any fixed compact subset $G \subset V$ we have

$$\phi(tK + x) = t^k \psi(K, x) + o(t^k)$$

uniformly in $K \subset G, x \in G$. Also $\psi(K, x)$ depends smoothly on x . Then $\psi(t(K + a) + x) = t^k \psi(K, ta + x) + o(t^k) = t^k \psi(K, x) + o(t^k)$. This proves the translation invariance of the limit.

It remains to prove surjectivity of Λ_k . We will need the following lemma.

3.1.6 LEMMA: *The map*

$$\Xi_k := \Lambda_k \circ \Theta'_{n-k}: C^\infty(V \times \mathbb{P}_+(V^*)^{n-k}, |\omega_V| \boxtimes L^{\boxtimes(n-k)}) \rightarrow C^\infty(V, Val_k(TV))$$

is an epimorphism.

Obviously Proposition 3.1.5(i) follows from Lemma 3.1.6.

Proof of Lemma 3.1.6: Let us describe Ξ_k explicitly. Let

$$\gamma = \mu \otimes h_{A_1} \otimes \cdots \otimes h_{A_{n-k}}$$

where h_{A_i} is the supporting functional of a set $A_i \in \mathcal{K}(V)$, $\mu \in C^\infty(V, |\omega_V|)$. Let $\mu = F(y)dy$ where dy is a Lebesgue measure. Then

$$(\Theta'_{n-k} \gamma)(K) = \frac{\partial^{n-k}}{\partial \lambda_1 \cdots \partial \lambda_{n-k}} \Big|_0 \int_{K + \sum_i \lambda_i A_i} F(y) dy.$$

Hence

$$\begin{aligned}
 (\Xi_k \gamma)(K) &= \lim_{r \rightarrow +0} \frac{1}{r^k} \frac{\partial^{n-k}}{\partial \lambda_1 \dots \partial \lambda_{n-k}} \Big|_0 \int_{rK+x+\sum_i \lambda_i A_i} F(y) dy \\
 &= \lim_{r \rightarrow +0} \frac{1}{r^k} \frac{\partial^{n-k}}{\partial \lambda_1 \dots \partial \lambda_{n-k}} \Big|_0 \left(F(x) \text{vol}(rK + \sum_i \lambda_i A_i) + o \left(\left(\sqrt{r^2 + \sum_i \lambda_i^2} \right)^n \right) \right) \\
 &= \lim_{r \rightarrow +0} \frac{1}{r^k} \frac{\partial^{n-k}}{\partial \lambda_1 \dots \partial \lambda_{n-k}} \Big|_0 F(x) \text{vol}(rK + \sum_i \lambda_i A_i) \\
 &= F(x) \frac{\partial^{n-k}}{\partial \lambda_1 \dots \partial \lambda_{n-k}} \Big|_0 \text{vol}(K + \sum_i \lambda_i A_i).
 \end{aligned}$$

This computation shows that Ξ_k is a morphism of $C^\infty(V)$ -modules. Let us denote

$$F_1 := C^\infty(\mathbb{P}_+(V^*)^{n-k}, |\wedge^n(V^*)| \otimes L^{\boxtimes(n-k)}).$$

It is easy to see that

$$C^\infty(V \times \mathbb{P}_+(V^*)^{n-k}, |\omega_V| \boxtimes L^{\boxtimes(n-k)}) = C^\infty(V, F_1).$$

Moreover

$$(\Xi_k f)(v) = \tilde{\Theta}_{n-k}(f(v)) \quad \forall v \in V, f \in F_1,$$

where

$$\tilde{\Theta}_{n-k}: F_1 \rightarrow \text{Val}_k^{\text{sm}}(V)$$

is the restriction of Θ'_{n-k} to the space F_1 which coincides with the subspace of $C^\infty(V \times \mathbb{P}_+(V^*)^{n-k}, |\omega_V| \boxtimes L^{\boxtimes(n-k)})$ consisting of elements invariant with respect to translations to vectors from V .

The map $\tilde{\Theta}_{n-k}$ commutes with the natural action of the group $GL(V)$. Hence $\tilde{\Theta}_{n-k}$ is onto by Irreducibility Theorem 1.2.3 and Casselman–Wallach Theorem 1.1.5. Proposition 1.1.8 implies that Ξ_k is onto as well. Thus Lemma 3.1.6 is proved. ■

3.1.7 COROLLARY: *The image of the map*

$$\Theta_k: \bigoplus_{i=0}^k C^\infty(V \times \mathbb{P}_+(V^*)^i, |\omega_V| \boxtimes L^{\boxtimes i}) \rightarrow SV(V)$$

is equal to W_{n-k} .

Proof: By Proposition 3.1.4 the image of the map Θ_k is contained in W_{n-k} . Let us prove the opposite inclusion by the induction in k . For $k = 0$ this is

just Proposition 3.1.2. Let us assume that $Im(\Theta_{k'}) = W_{n-k'}$ for $k' < k$. Let $\phi \in W_{n-k}$. By Proposition 3.1.5 there exists $\psi \in Im(\Theta_k)$ such that $\Lambda_{n-k}(\phi) = \Lambda_{n-k}(\psi)$. It follows that $\phi - \psi \in W_{n-k+1}$. Applying the induction assumption to this valuation we obtain the result. ■

3.1.8 COROLLARY: *Polynomial valuations from W_k are dense in W_k . Polynomial valuations from $QV(V)$ are dense in $QV(V)$.*

Proof: First notice that the second statement follows from the first one. Indeed it is true since $W_0 = SV(V)$ is dense in $QV(V)$. The first statement follows from Corollary 3.1.7 and the obvious fact that the image under Θ_k of any element of $C^\infty(V \times \mathbb{P}_+(V^*)^i, |\omega_V| \boxtimes L^{\boxtimes i})$ which is polynomial with respect to translations in V , is a polynomial valuation. ■

From this corollary we immediately get

3.1.9 COROLLARY: *Let G be a compact subgroup of $GL(V)$. Then G -invariant polynomial valuations are dense in the space of G -invariant quasi-smooth valuations.*

Let us now introduce another decreasing filtration on $SV(V)$. Set

$$\gamma_i := \{\phi \in SV(V) \mid \phi(K) = 0 \text{ if } \dim K < i\}.$$

Clearly $SV(V) = \gamma_0 \supset \gamma_1 \supset \dots \supset \gamma_n \supset \gamma_{n+1} = 0$.

3.1.10 THEOREM: (i) $W_1 = \gamma_1$.

(ii) $\gamma_{j+1} \subset W_j \subset \gamma_j$ for any j .

Proof: (i) First note that for any j we have $W_j \subset \gamma_j$. This follows from Proposition 3.1.2 applied for $(j - 1)$ -dimensional subsets. Let us prove that $\gamma_1 \subset W_1$. Let $\phi \in \gamma_1$, i.e. ϕ vanishes on points. Hence for any $K \in \mathcal{K}(V)$ the function $[0, 1] \rightarrow C^n(V)$ given by $t \mapsto [x \mapsto \phi(tK + x)]$ vanishes at $t = 0$. Hence $\phi \in W_1$ by the definition of W_1 .

(ii) We have proven the second inclusion in (ii). Let us prove the first one, namely $\gamma_{j+1} \subset W_j$. Assume that this is not true. Then there exists $l < j$ and $\phi \in \gamma_{j+1} \cap W_l$ such that $\phi \notin W_{l+1}$. Set $\tilde{\phi} := \Lambda_l(\phi) \in C^\infty(V, Val_l^{sm}(TV))$. Then $\tilde{\phi} \neq 0$ by Proposition 3.1.5(ii). From the construction of Λ_l it follows that at each point $x \in V$ one has $\tilde{\phi}_x \in Val_l^{sm}(T_x V) \cap \gamma_{j+1}$. But the last intersection vanishes; for translation invariant valuations this was proved in [4] (see the beginning of Section 3 in [4]). Thus we get a contradiction. ■

4. The multiplicative structure

In this section we construct a canonical multiplicative structure on $SV(V)$. $SV(V)$ will become a commutative associative algebra with unit (where the unit is the Euler characteristic).

First of all we will construct for any linear spaces X and Y the exterior product

$$SV(X) \times SV(Y) \rightarrow QV(X \times Y),$$

which is a bilinear continuous map.

First let us introduce some notation. Let us denote for brevity

$$F_X := \bigoplus_{k=0}^{\dim X} C^\infty(X \times \mathbb{P}_+(X^*)^k, |\omega_X| \boxtimes L^{\boxtimes k}),$$

$$F_Y := \bigoplus_{l=0}^{\dim Y} C^\infty(Y \times \mathbb{P}_+(Y^*)^l, |\omega_Y| \boxtimes L^{\boxtimes l}).$$

First, for any $k \leq \dim X, l \leq \dim Y$ let us define a multilinear map

$$M: C^\infty(X, |\omega_X|) \times C^\infty(\mathbb{P}_+(X^*), L)^k \times C^\infty(Y, |\omega_Y|) \times C^\infty(\mathbb{P}_+(Y^*), L)^l \rightarrow QV(X \times Y).$$

Let

$$\mu \in C^\infty(X, |\omega_X|), \nu \in C^\infty(Y, |\omega_Y|), \xi_i \in C^\infty(\mathbb{P}_+(X^*), L), \eta_j \in C^\infty(\mathbb{P}_+(Y^*), L)$$

where $i = 1, \dots, k, j = 1, \dots, l$. First let us assume that $\xi_i = h_{A_i}$ is a supporting functional of a convex set $A_i \in \mathcal{K}(X)$, and $\eta_j = h_{B_j}$ is a supporting functional of $B_j \in \mathcal{K}(Y)$. Let us define

$$M(\mu, \xi_1, \dots, \xi_k; \nu, \eta_1, \dots, \eta_l)(K) = \frac{\partial^k}{\partial \lambda_1 \dots \partial \lambda_k} \frac{\partial^l}{\partial \theta_1 \dots \partial \theta_l} |_0(\mu \boxtimes \nu) \left(K + \sum_{i=1}^k \lambda_i (A_i \times 0) + \sum_{j=1}^l \theta_j (0 \times B_j) \right).$$

It is clear that the right hand side is Minkowski additive with respect to A_i and B_j . Hence using the same argument as in the construction of Θ'_i (in Section 2.2) and using Theorem 2.2.1 we extend M to a continuous multilinear functional defined for all $\xi_i \in C^\infty(\mathbb{P}_+(X^*), L), \eta_j \in C^\infty(\mathbb{P}_+(Y^*), L)$.

Hence by the L. Schwartz kernel theorem we get a bilinear continuous map

$$M: F_X \times F_Y \rightarrow QV(X \times Y).$$

Recall that we have canonical surjections

$$\Theta_X: F_X \rightarrow SV(X), \quad \Theta_Y: F_Y \rightarrow SV(Y),$$

where now we use the subscript to emphasize dependence on the space.

4.1.1 LEMMA: *The bilinear map $M: F_X \times F_Y \rightarrow QV(X \times Y)$ admits a unique factorization to a continuous bilinear map*

$$M': SV(X) \times SV(Y) \rightarrow QV(X \times Y)$$

such that $M = M' \circ (\Theta_X \times \Theta_Y)$.

Proof: The uniqueness of such a factorization is obvious due to the surjectivity of Θ_X, Θ_Y . Let us prove existence. Let us fix $f \in Ker\Theta_X$. It is enough to show that $M(f, g) = 0$ for any $g \in F_Y$. It is enough to assume that $g = \nu \otimes \eta_1 \otimes \dots \otimes \eta_p$ where $\nu \in C^\infty(Y, |\omega_Y|), \eta_i \in C^\infty(\mathbb{P}_+(Y^*), L)$, and moreover $\eta_i = h_{B_i}$ where $B_i \in \mathcal{K}(Y)$.

Let us prove that for any $w \in F_X$ and $K \in \mathcal{K}(X \times Y)$ one has

$$(3) \quad M(w, g)(K) = \frac{\partial^p}{\partial \theta_1 \dots \partial \theta_p} \Big|_0 \int_{y \in Y} \Theta_X(w) \left(\left(K + \sum_{j=1}^p \theta_j (0 \times B_j) \right) \cap (X \times \{y\}) \right) d\nu(y).$$

Note that this identity implies Lemma 4.1.1.

Let us check the identity (3). First let us check it for $w = \mu \otimes \xi_1 \otimes \dots \otimes \xi_k$ where $\xi_i = h_{A_i}, A_i \in \mathcal{K}(X)$. For such w , using Theorem 2.2.1 one obtains

$$\begin{aligned} M(w, g) &= \frac{\partial^k}{\partial \lambda_1 \dots \partial \lambda_k} \frac{\partial^p}{\partial \theta_1 \dots \partial \theta_p} \Big|_0 (\mu \boxtimes \nu) \left(K + \sum_{i=1}^k \lambda_i (A_i \times 0) + \sum_{j=1}^p \theta_j (0 \times B_j) \right) \\ &= \frac{\partial^p}{\partial \theta_1 \dots \partial \theta_p} \frac{\partial^k}{\partial \lambda_1 \dots \partial \lambda_k} \Big|_0 \int_{y \in Y} \mu \left(\left(\left(K + \sum_{j=1}^p \theta_j (0 \times B_j) \right) \cap (X \times \{y\}) \right) \right. \\ &\quad \left. + \sum_{i=1}^k \lambda_i A_i \right) d\nu(y) \\ &= \frac{\partial^p}{\partial \theta_1 \dots \partial \theta_p} \Big|_0 \int_{y \in Y} \Theta_X(w) \left(\left(K + \sum_{j=1}^p \theta_j (0 \times B_j) \right) \cap (X \times \{y\}) \right) d\nu(y). \end{aligned}$$

Let us return now to the case of general $w \in F_X$. To prove the equality (3) it remains to show that the right hand side of (3) is continuous with respect to

$w \in F_X$ for fixed $K \in \mathcal{K}(X \times Y)$ and $B_1, \dots, B_p \in \mathcal{K}(Y)$. Clearly, it is enough to prove the continuity with respect to $w \in C^\infty(X \times \mathbb{P}_+(X^*)^k, |\omega_X| \boxtimes L^{\boxtimes k})$ for any $k = 0, 1, \dots, n$.

Let us fix a large compact subset $G \subset X$ containing the projection of K to X in its interior. By Lemma 1.1.7 there exist a constant C , a compact subset $\tilde{G} \subset X$ containing G , and an integer N such that

$$w = \sum_{s=1}^{\infty} \mu_s \otimes h_1^s \otimes \dots \otimes h_k^s,$$

where $\mu_s \in C^\infty(X, |\omega_X|)$, $h_i^s \in C^\infty(\mathbb{P}_+(X^*), L)$ and

$$(4) \quad \sum_{s=1}^{\infty} \|\mu_s\|_{C^{k+p+2}(G)} \prod_{i=1}^k \|h_i^s\|_{C^{k+p+2}(\mathbb{P}_+(X^*))} \leq C \|w\|_{C^N(\tilde{G} \times \mathbb{P}_+(X^*)^k)}.$$

Adding to and subtracting from each h_i^s a supporting functional of the unit Euclidean ball times a constant depending on $\|h_i^s\|_{C^2(\mathbb{P}_+(X^*))}$, we may assume that $h_i^s = h_{A_i^s}$ is a supporting functional of a convex compact set A_i^s . Thus

$$w = \sum_{s=1}^{\infty} \mu_s \otimes h_{A_1^s} \otimes \dots \otimes h_{A_k^s}$$

and

$$(5) \quad \sum_{s=1}^{\infty} \|\mu_s\|_{C^{k+p+2}(G)} \prod_{i=1}^k \|h_{A_i^s}\|_{C^{k+p+2}(\mathbb{P}_+(X^*))} \leq C \|w\|_{C^N(\tilde{G} \times \mathbb{P}_+(X^*)^k)}.$$

Theorem 2.2.1 implies also that there exist a constant C' , depending on the B_j and G , and a compact subset $G' \subset Y$ such that

$$(6) \quad \left\| \frac{\partial^k}{\partial \lambda_1 \dots \partial \lambda_k} |_0 (\mu \boxtimes \nu) \left(K + \sum_{i=1}^k \lambda_i (A_i \times 0) + \sum_{j=1}^p \theta_j (0 \times B_j) \right) \right\|_{C^p[0,1]^p}$$

$$(7) \quad \leq C' \|\nu\|_{C^{k+p+1}(G')} \cdot \|\mu\|_{C^{k+p+1}(G)} \prod_{i=1}^k \|h_{A_i}\|_{C^{k+p+1}}$$

where the function in the left hand side of the inequality is considered as a function of $(\theta_1, \dots, \theta_p) \in [0, 1]^p$. Hence the function

$$(8) \quad (\theta_1, \dots, \theta_p) \mapsto \sum_{s=1}^{\infty} \frac{\partial^k}{\partial \lambda_1 \dots \partial \lambda_k} |_0 (\mu_s \boxtimes \nu) \left(K + \sum_{i=1}^k \lambda_i (A_i^s \times 0) + \sum_{j=1}^p \theta_j (0 \times B_j) \right)$$

$$(9) \quad = \int_{y \in Y} \Theta_X(w) \left(\left(K + \sum_{j=1}^p \theta_j (0 \times B_j) \right) \cap (X \times \{y\}) \right) d\nu(y)$$

belongs to $C^p[0, 1]^p$, and using (5) and (6) the sum of its C^p -norms if the summands in (8) can be estimated from above by

$$\begin{aligned}
 C' \|\nu\|_{C^{k+p+1}(G')} \cdot \sum_s \|\mu_s\|_{C^{k+p+1}(G)} \prod_{i=1}^k \|h_{A_i^s}\|_{C^{k+p+1}} \\
 \leq C' C \|\nu\|_{C^{k+p+1}(G')} \|w\|_{C^N(\tilde{G})}.
 \end{aligned}$$

Equality (3) is proved, and hence Lemma 4.1.1 follows. ■

For any $\phi \in SV(X), \psi \in SV(Y)$ we will denote $M'(\phi, \psi)$ by $\phi \boxtimes \psi$ and call it the exterior product of ϕ and ψ . In [4] we have defined the exterior product of polynomial smooth valuations. The point of this construction is that it extends to smooth valuations without any assumption of polynomiality.

Let us define now the product on $SV(V)$. Let $\Delta: V \hookrightarrow V \times V$ be the diagonal imbedding. For $\phi, \psi \in SV(V)$ set

$$\phi \cdot \psi := \Delta^*(\phi \boxtimes \psi)$$

where Δ^* denotes the restriction of a valuation on $V \times V$ to the diagonal.

4.1.2 THEOREM: (1) For $\phi, \psi \in SV(V)$ the product $\phi \cdot \psi \in SV(V)$.

(2) The product $SV(V) \times SV(V) \rightarrow SV(V)$ is continuous.

(3) Equipped with this multiplication, $SV(V)$ becomes an associative commutative unital algebra when the unit is the Euler characteristic.

(4) The filtration $\{W_\bullet\}$ is compatible with this multiplication, i.e.

$$W_i \cdot W_j \subset W_{i+j}.$$

Proof: To prove (1) notice first of all that $\Delta^*: QV(V \times V) \rightarrow QV(V)$. Hence $SV(V) \cdot SV(V) \subset QV(V)$. But since the product commutes with the action of $Aff(V)$, the product of $Aff(V)$ -smooth vectors is $Aff(V)$ -smooth. Hence $SV(V) \cdot SV(V) \subset SV(V)$. The continuity of the product follows by the same reason.

Let us prove (3) and (4). Using Corollary 3.1.8 they reduce to the case of polynomial valuations. But for polynomial valuations the corresponding statements were proved in [4]. ■

Let us now describe the associated graded algebra $gr_W SV(V)$ with respect to the filtration $\{W_i\}$. Recall that $gr_W SV(V) := \bigoplus_{i=0}^n W_i/W_{i+1}$, and it carries the natural algebra structure.

4.1.3 THEOREM: *The associated graded algebra $gr_W SV(V)$ is canonically isomorphic to the graded algebra $C^\infty(V, Val^{sm}(V))$ with the pointwise multiplication on V and the k -th graded term of it is equal to $C^\infty(V, Val_k^{sm}(V))$.*

Proof: First let us recall that the isomorphism W_k/W_{k+1} with $C^\infty(V, Val_k(TV))$ is induced by the map $\Lambda_k: W_k \rightarrow C^\infty(V, Val_k(TV))$ defined in Section 3. Let $\phi \in W_k$. We have $(\Lambda_k(\phi))(x)(K) = \lim_{r \rightarrow +0} r^{-k} \phi(rK + x)$. Thus the isomorphism of vector spaces follows from Proposition 3.1.5. Now it remains to check that this map is a homomorphism of algebras. By Corollary 3.1.8 the result reduces to the case of polynomial valuations. But for polynomial valuations the result was proved in [4]. ■

5. Integration with respect to the normal cycle

In Subsection 5.1 we fix some notation and summarize known relevant facts about construction of valuations using integration with respect to the normal cycle. The main new results of this section are contained in Subsection 5.2. These are Theorems 5.2.1 and 5.2.2 about construction of smooth valuations using integration with respect to the normal cycle.

5.1 MAIN CONSTRUCTION AND ITS PROPERTIES. Let V be a real vector space of dimension n . Then clearly $T^*V = V \times V^*$. Let $K \in \mathcal{K}(V)$. Let $x \in K$.

5.1.1 Definition: A tangent cone to K at x is a set denoted by $T_x K$ which is equal to the closure of the set $\{y \in V | \exists \varepsilon > 0 \ x + \varepsilon y \in K\}$.

It is easy to see that $T_x K$ is a closed convex cone.

5.1.2 Definition: A normal cone to K at x is the set

$$Nor_x K := \{y \in V^* | y(x) \geq 0 \forall x \in T_x K\}.$$

Thus $Nor_x K$ is also a closed convex cone.

5.1.3 Definition: Let $K \in \mathcal{K}(V)$. The **characteristic cycle** of K is the set

$$CC(K) := \bigcup_{x \in K} Nor_x(K).$$

5.1.4 Remark: The notion of the characteristic cycle is not new. First an almost equivalent notion of normal cycle (see below) was introduced by Wintgen [19], and then studied further by Zähle [20] by the tools of geometric measure theory. Characteristic cycles of subanalytic sets of real analytic manifolds were introduced by Kashiwara (see [12], Chapter 9) using the tools of sheaf theory, and independently by J. Fu [8] using rather different tools of geometric measure theory. The elementary approach described above is sufficient for the purposes of this article.

It is easy to see that $CC(K)$ is a closed n -dimensional subset of $T^*V = V \times V^*$ invariant with respect to the multiplication by non-negative numbers acting on the second factor. Sometimes we will also use the following notation. Let $\underline{0}$ denote the zero section of T^*V , i.e. $\underline{0} = V \times \{0\}$. Set

$$\begin{aligned} \underline{CC}(K) &:= CC(K) \setminus \underline{0}, \\ \tilde{CC}(K) &:= \underline{CC}/\mathbb{R}_{>0}. \end{aligned}$$

Thus $\tilde{CC}(K) \subset \mathbb{P}_+(T^*V)$. Let us denote by $N(K)$ the image of $\tilde{CC}(K)$ under the involution on $\mathbb{P}_+(T^*V)$ of the change of an orientation of a line. $N(K)$ is called the **normal cycle** of K .

Let us denote by

$$p: T^*V \rightarrow V$$

the canonical projection. Let us denote by o the orientation bundle of V . Note that a choice of orientation on V induces canonically an orientation on $CC(K)$ and $N(K)$ for any $K \in \mathcal{K}(V)$. Let us denote by $\tilde{C}^1(T^*V, \Omega^n \otimes p^*o)$ the space of C^1 -smooth sections of $\Omega^n \otimes p^*o$ over T^*V such that the restriction of p to the support of this section is proper.

5.1.5 THEOREM: *For any $\omega \in \tilde{C}^1(T^*V, \Omega^n \otimes p^*o)$, the map $\mathcal{K}(V) \rightarrow \mathbb{C}$ given by $K \mapsto \int_{CC(K)} \omega$ defines a continuous valuation on $\mathcal{K}(V)$.*

The proof of this result can be found in [6]. However, for a special choice of the form ω leading to the curvature measures this theorem was proved much earlier by M. Zähle [20].

We immediately obtain the following corollary.

5.1.6 COROLLARY: *For any $\eta \in C^1(\mathbb{P}_+(T^*V), \Omega^{n-1} \otimes p^*o)$, the map $\mathcal{K}(V) \rightarrow \mathbb{C}$ given by $K \mapsto \int_{N(K)} \eta$ defines a continuous valuation on $\mathcal{K}(V)$.*

We will also need the following statement.

5.1.7 THEOREM ([6]): *The map*

$$\mathcal{K}(V) \times (C^1(V, |\omega_V|) \oplus C^1(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o)) \rightarrow \mathbb{C}$$

given by

$$(K, (\omega, \eta)) \mapsto \int_K \omega + \int_{N(K)} \eta$$

is continuous.

Theorem 5.1.7 immediately implies the following corollary.

5.1.8 COROLLARY: (i) *The map $C^1(V, |\omega_V|) \oplus C^1(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o) \rightarrow CV(V)$ given by $(\omega, \eta) \mapsto [K \mapsto \int_K \omega + \int_{N(K)} \eta]$ is continuous.*

(ii) *For any compact set $G \subset V$ there exist a larger compact set $\tilde{G} \subset V$ and a constant $C = C(G)$ such that for any*

$$(\omega, \eta) \in C^1(V, |\omega_V|) \oplus C^1(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o)$$

one has

$$\sup_{K \subset G, K \in \mathcal{K}(V)} \left| \int_K \omega + \int_{N(K)} \eta \right| \leq C(\|\omega\|_{C^1(\tilde{G})} + \|\eta\|_{C^1(p^{-1}\tilde{G})}).$$

5.1.9 PROPOSITION: (i) *For any*

$$(\omega, \eta) \in C^\infty(V, |\omega_V|) \oplus C^\infty(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o)$$

the valuation $[K \mapsto \int_K \omega + \int_{N(K)} \eta]$ is smooth, i.e. belongs to $SV(V)$.

(ii) *The induced map*

$$C^\infty(V, |\omega_V|) \oplus C^\infty(\mathbb{P}_+(V^*), \Omega^{n-1} \otimes p^*o) \rightarrow SV(V)$$

is continuous.

Proof: Since the construction of integration with respect to the normal cycle is equivariant with respect to the natural action of the group $GL(V)$ on all spaces, it is sufficient to prove the proposition with $SV(V)$ replaced with $QV(V)$ everywhere. For simplicity we will ignore the summand $\int_K \omega$. The last case is simpler and it can be considered similarly.

For any $(x, t) \in V \times [0, 1]$ let us define the map $\tau_{(x,t)}: \mathbb{P}_+(T^*V) \rightarrow \mathbb{P}_+(T^*V)$ by

$$\tau_{(x,t)}((y, n)) := (ty + x, n).$$

Then we have

$$\int_{N(tK+x)} \eta = \int_{N(K)} \tau_{(x,t)}^* \eta.$$

Clearly the form $\tau_{(x,t)}^* \eta$ depends smoothly on (x, t) . This implies part (i) of the proposition.

Let us prove part (ii). Let us fix a compact set $G \subset V$ and $N \in \mathbb{N}$. We have

$$\begin{aligned} & \sup_{x \in G, K \subset G} \left\| t \mapsto \int_{N(tK+x)} \eta \right\|_{C^N[0,1]} = \\ & \sup_{x \in G, K \subset G} \left\| t \mapsto \int_{N(K)} \tau_{(x,t)}^* \eta \right\|_{C^N[0,1]} \leq C \|\eta\|_{C^{N+1}(p^{-1}\tilde{G})}, \end{aligned}$$

where C and \tilde{G} are from Corollary 5.1.8(ii). This implies Proposition 5.1.9. ■

5.2 MAIN RESULTS. Let us denote by $C_{tr}^\infty(\mathbb{P}_+(T^*V), \Omega^{n-1} \otimes p^*o)$ the subspace consisting of elements of the space $C^\infty(\mathbb{P}_+(T^*V), \Omega^{n-1} \otimes p^*o)$ which are invariant under translations with respect to vectors from V . Elements of this space define translation invariant smooth valuations.

5.2.1 THEOREM: *Consider the map*

$$\mathbb{C} \cdot vol_V \oplus C_{tr}^\infty(\mathbb{P}_+(T^*V), \Omega^{n-1} \otimes p^*o) \rightarrow Val^{sm}(V)$$

given by $(\omega, \eta) \mapsto [K \mapsto \int_K \omega + \int_{N(K)} \eta]$. This map is onto.

Proof: Clearly this map commutes with the natural action of $GL(V)$ on both spaces. It is easy to see that the image of this map intersects non-trivially each subspace $Val_i^{ev/odd}$ for $i = 0, 1, \dots, n$. Hence by Irreducibility Theorem 1.2.3 the image of this map is dense in $Val^{sm}(V)$. By Casselman–Wallach Theorem 1.1.5 the image of this map is closed. Hence it coincides with $Val^{sm}(V)$. ■

5.2.2 THEOREM: *The map*

$$C^\infty(V, |\omega_V|) \oplus C^\infty(\mathbb{P}_+(T^*V), \Omega^{n-1} \otimes p^*o) \rightarrow SV(V)$$

is onto.

In order to prove this theorem we will introduce a decreasing filtration on the space $C^\infty(V, |\omega_V|) \oplus C^\infty(\mathbb{P}_+(T^*V), \Omega^{n-1} \otimes p^*o)$ and show that it maps onto the filtration W_\bullet on $SV(V)$. Let us start with some general considerations.

Let X be a smooth manifold. Let $p: P \rightarrow X$ be a smooth bundle. Let $\Omega^N(P)$ be the vector bundle over P of N -forms. Let us introduce a filtration of $\Omega^N(P)$ by vector subbundles $W_i(P)$ as follows. For every $y \in P$ set

$$(W_i(P))_y := \{\omega \in \wedge^N T_y^* P \mid \omega|_F \equiv 0 \text{ for all } F \in Gr_N(T_y P) \\ \text{with } \dim(F \cap T_y(p^{-1}p(y))) > N - i\}.$$

Clearly we have

$$\Omega^N(P) = W_0(P) \supset W_1(P) \supset \dots \supset W_N(P) \supset W_{N+1}(P) = 0.$$

We will study this filtration in greater detail.

Let us make some elementary observations from linear algebra. Let L be a finite dimensional vector space. Let $E \subset L$ be a linear subspace. For a non-negative integer i set

$$W(L, E)_i := \{\omega \in \wedge^N L^* \mid \omega|_F \equiv 0 \text{ for all } F \subset L \text{ with } \dim(F \cap E) > N - i\}.$$

Clearly

$$\wedge^N L^* = W(L, E)_0 \supset W(L, E)_1 \supset \dots \supset W(L, E)_N \supset W(L, E)_{N+1} = 0.$$

5.2.3 LEMMA: *There exists a canonical isomorphism of vector spaces*

$$W(L, E)_i / W(L, E)_{i+1} = \wedge^{N-i} E^* \otimes \wedge^i (L/E)^*.$$

Proof: First note that for every $0 \leq j \leq N$ we have canonical map

$$\wedge^j E^\perp \otimes \wedge^{N-j} L^* \rightarrow \wedge^N L^*$$

given by $x \otimes y \mapsto x \wedge y$. It is easy to see that

$$W(L, E)_i = \text{Im} \left[\bigoplus_{j \geq i} (\wedge^j E^\perp \otimes \wedge^{N-j} L^*) \rightarrow \wedge^N L^* \right].$$

Note that the induced map

$$\wedge^i E^\perp \otimes \wedge^{N-i} L^* \rightarrow W(L, E)_i / W(L, E)_{i+1}$$

is surjective and factorizes as follows (using the equality $E^\perp = (L/E)^*$ and the canonical map $L^* \rightarrow E^*$):

$$\begin{array}{ccc} \wedge^i E^\perp \otimes \wedge^{N-i} L^* & \longrightarrow & W(L, E)_i / W(L, E)_{i+1} \\ & \searrow & \nearrow \\ & \wedge^i (L/E)^* \otimes \wedge^{N-i} E^* & \end{array}$$

Let us check that the obtained map

$$\wedge^i(L/E)^* \otimes \wedge^{N-i}E^* \rightarrow W(L, E)_i/W(L, E)_{i+1}$$

is an isomorphism. Let us fix a splitting $L = E \oplus F$. Then

$$W(L, E)_i \simeq \bigoplus_{j \geq i} \wedge^j F^* \otimes \wedge^{N-j} E^*.$$

Hence $W(L, E)_i/W(L, E)_{i+1} \simeq \wedge^i F^* \otimes \wedge^{N-i} E^* \simeq \wedge^i(L/E)^* \otimes \wedge^{N-i} E^*$. ■

Let us apply the above construction to the case $P = \mathbb{P}_+(T^*X)$ with X being a smooth manifold of dimension n . The above construction defines a filtration of the vector bundle $\Omega^{n-1}(P)$ by vector subbundles. Twisting by the pullback p^*o of the orientation sheaf o of X we obtain a filtration $\{W_\bullet(P)\}$ by vector subbundles of the vector bundle $\Omega^{n-1}(P) \otimes p^*o$:

$$\Omega^{n-1}(P) \otimes p^*o = W_0(P) \supset W_0(P) \supset W_1(P) \supset \dots \supset W_{n-1}(P).$$

Let us denote by $\Omega_{P/X}^{n-1}(P)$ the vector bundle over P of differential forms along the fibers. (Thus $\Omega_{P/X}^{n-1}(P)$ is the quotient bundle of $\Omega^{n-1}(P)$.)

5.2.4 LEMMA: For $0 \leq i \leq n - 1$ there exists a canonical isomorphism

$$(10) \quad J_i: W_i(P)/W_{i+1}(P) \xrightarrow{\sim} \Omega_{P/X}^{n-1-i} \otimes p^*(\wedge^i T^*X) \otimes p^*o.$$

Proof: This is an immediate corollary of Lemma 5.2.3. ■

Recall that we denote by $Val_i^{sm}(TX)$ the (infinite dimensional) vector bundle over X whose fiber over a point $x \in X$ is equal to the space of translation invariant i -homogeneous $GL_n(\mathbb{R})$ -smooth valuations on $T_x X$. For any point $x \in X$ we have the canonical map

$$(11) \quad C^\infty(\Omega^{n-1-i}(\mathbb{P}_+(T_x^*X))) \otimes \wedge^i T_x^*X \otimes o_{T_x^*X} \rightarrow Val_i^{sm}(T_x X)$$

where $o_{T_x^*X}$ denotes the orientation sheaf of T_x^*X (this map is given by integration over a normal cycle). This map induces a continuous linear map

$$(12) \quad \Psi_i: C^\infty(P, \Omega_{P/X}^{n-1-i}(P) \otimes p^*(\wedge^i T^*X) \otimes p^*o) \rightarrow C^\infty(X, Val_i^{sm}(TX)).$$

Let us now apply these constructions to an affine space V (instead of X). We will identify $P := \mathbb{P}_+(T^*V) \rightarrow V$ with $\mathbb{P}_+(V^*) \times V$. Then the projection $p: P = \mathbb{P}_+(V^*) \times V$ is the projection to the second factor. Consider the following map:

$$\Xi: C^\infty(V, |\omega_V|) \oplus C^\infty(P, \Omega^{n-1}(P) \otimes p^*o) \rightarrow SV(V),$$

which is given by

$$\Xi((\nu, \eta))(K) = \nu(K) + \int_{N(K)} \eta.$$

5.2.5 PROPOSITION:

$$\begin{aligned} \Xi(C^\infty(V, |\omega_V|)) &= W_n \\ \Xi(C^\infty(P, W_i(P)) \oplus C^\infty(V, |\omega_V|)) &= W_i, \quad i = 0, 1, \dots, n-1, \end{aligned}$$

where W_i in the right hand side denotes the i -th term of the filtration on $SV(V)$.

Proof: The statement is obvious for $i = n$. Assume that $i < n$. Let us fix for simplicity of notation an orientation on V . Thus the orientation sheaf o becomes trivialized. First let us show that

$$(13) \quad \Xi(C^\infty(P, W_i(P))) \subset W_i.$$

Fix any $\omega \in C^\infty(P, W_i(P))$. We have to show that for any $K \in \mathcal{K}(V)$ and any $x \in V$

$$\int_{N(tK+x)} \omega = O(t^i) \quad \text{as } t \rightarrow +0.$$

This easily follows from the fact that any such ω belongs to the space

$$\bigoplus_{j=i}^{n-1} C^\infty(P, \Omega_{P/V}^{n-1-j}(P)) \otimes \wedge^j V^*$$

Thus the inclusion (13) is proved. Hence we obtain a map

$$(14) \quad \Xi_i: C^\infty(P, W_i(P)/W_{i+1}(P)) \rightarrow W_i/W_{i+1}.$$

We will show that Ξ_i is surjective. This will imply Proposition 5.2.5 by the induction in i . Recall that by Proposition 3.1.5 we have canonical isomorphism

$$(15) \quad I_i: W_i/W_{i+1} \xrightarrow{\sim} C^\infty(V, Val_i^{sm}(V)).$$

One has the following lemma.

5.2.6 LEMMA: *The following diagram is commutative:*

$$(16) \quad \begin{array}{ccc} C^\infty(P, W_i(P)/W_{i+1}(P)) & \xrightarrow{\Xi_i} & W_i/W_{i+1} \\ \downarrow J_i & & \downarrow I_i \\ C^\infty(P, \Omega_{P/X}^{n-1-i}(P) \otimes \wedge^i V^*) & \xrightarrow{\Psi_i} & C^\infty(V, Val_i^{sm}(V)) \end{array}$$

where the maps J_i, Ψ_i, Ξ_i, I_i are defined by (10), (12), (14), (15) respectively.

Let us postpone the proof of Lemma 5.2.6 and finish the proof of Proposition 5.2.5. Since J_i and I_i are isomorphisms it is enough to prove surjectivity of Ψ_i . Let us consider the Fréchet spaces

$$F_1 := C^\infty(\mathbb{P}_+(V^*), \Omega^{n-1-i}(\mathbb{P}_+(V^*)) \otimes \wedge^i V^*),$$

$$F_2 := Val_i^{sm}(V).$$

Then clearly

$$C^\infty(P, \Omega_{P/X}^{n-1-i}(P) \otimes \wedge^i V^*) = C^\infty(V, F_1),$$

$$C^\infty(V, Val_i^{sm}) = C^\infty(V, F_2).$$

By (11) we have the canonical map

$$f_i: F_1 \rightarrow F_2.$$

Clearly, for any $\psi \in C^\infty(V, F_1)$ and any $y \in V$ one has

$$(\Psi_i \psi)(y) = f_i(\psi(y)).$$

Moreover, f_i is surjective by Irreducibility Theorem 1.2.3 and Casselman–Wallach Theorem 1.1.5. Hence Ψ_i is surjective by Proposition 1.1.8. ■

Thus it remains to prove Lemma 5.2.6.

Proof of Lemma 5.2.6: Recall that for $\phi \in W_i/W_{i+1}$ and for all $x \in V$, $K \in \mathcal{K}(V)$ one has

$$(I_i \phi)(x, K) = \lim_{r \rightarrow +0} \frac{1}{r^i} \phi(rK + x).$$

Let us fix $\eta \in C^\infty(P, W_i(P)/W_{i+1}(P))$. Let us fix a basis e_1^*, \dots, e_n^* in V^* . Then we can write

$$J_i(\eta) = \sum_{j_1, \dots, j_i} \eta_{j_1, \dots, j_i} \otimes e_{j_1}^* \wedge \dots \wedge e_{j_i}^*$$

where $\eta_{j_1, \dots, j_i} \in C^\infty(P, \Omega_{P/X}^{n-1-i}(P))$. Then

$$\begin{aligned} (I_i(\Xi_i \eta))(x, K) &= \sum_{j_1, \dots, j_i} \lim_{r \rightarrow +0} \frac{1}{r^i} \int_{N(rK+x)} \eta_{j_1, \dots, j_i} \otimes e_{j_1}^* \wedge \dots \wedge e_{j_i}^* \\ &= \sum_{j_1, \dots, j_i} \int_{N(K)} \eta_{j_1, \dots, j_i}|_{p^{-1}(x)} \otimes e_{j_1}^* \wedge \dots \wedge e_{j_i}^* \\ &= (\Psi_i(J_i \eta))(x, K). \quad \blacksquare \end{aligned}$$

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